where T_m^n is the truncation error.

Using the relations

$$(1-\gamma_1)U_m^n+\gamma_1 U_m^{n+1} = U_m^n+\gamma_1 k\,\frac{\partial U_m^n}{\partial t}+0(k^2),$$

$$\frac{1}{h}\,\mu_x\delta_x[(1-\gamma_1)\,U_m^n+\gamma_1\,U_m^{n+1}]$$

$$=\frac{1}{h}\,\mu_x\delta_x\left[\,U_m^n+\gamma_1 k\frac{\partial U_m^n}{\partial t}+0(k^2)\right]$$

$$=\frac{\partial U_m^n}{\partial x}+\frac{h^2}{6}\,\frac{\partial^3 U_m^n}{\partial x^3}+\gamma_1 k\,\frac{\partial^2 U_m^n}{\partial t\partial x}+0(h^2k),$$

$$\frac{1}{h^2}\delta_x^2[(1+\gamma_1)U_m^n+\gamma_1 U_m^{n+1}]$$

$$=\frac{1}{h^2}\delta_x^2\left[\,U_m^n+\gamma_1 k\frac{\partial U_m^n}{\partial t}+0(k^2)\right]$$

$$=\frac{\partial^2 U_m^n}{\partial x^2}+\frac{1}{12}\,h^2\frac{\partial^4 U_m^n}{\partial x^4}+\gamma_1 k\frac{\partial^3 U_m^n}{\partial t\partial x^2}+0(kh^2) \quad (5.331)$$

into the right side of (5.330) and expanding by Taylor series, we get

$$\phi\left[x_m,\, t_n,\, U_m^n,\, \frac{\partial U_m^n}{\partial t},\, \frac{\partial U_m^n}{\partial x},\, \frac{\partial^2 U_m^n}{\partial x^2}\right]+\left[\gamma_1 k\phi_t+\gamma_1 k\,\frac{\partial U_m^n}{\partial t}\phi_u+\tfrac{1}{2}k\,\frac{\partial^2 U_m^n}{\partial t^2}\phi_{u_t}\right.$$

$$\left.+\left(\frac{h^2}{6}\,\frac{\partial^3 U_m^n}{\partial x^3}+\gamma_1 k\frac{\partial^2 U_m^n}{\partial t\,\partial x}\right)\phi_{u_x}+\left(\frac{1}{12}\,h^2\frac{\partial^4 U_m^n}{\partial x^4}+\gamma_1 k\frac{\partial^3 U_m^m}{\partial t\partial x^2}\right)\phi_{u_{xx}}\right]$$

$$+0(k^2+h^4) = T_m^n \quad (5.332)$$

In view of the relations

$$\phi\left(x_m,\, t_n,\, U_m^n,\, \frac{\partial U_m^n}{\partial t},\, \frac{\partial U_m^n}{\partial x},\, \frac{\partial^2 U_m^n}{\partial x^2}\right) = 0$$

and

$$-\phi_{u_t}u_{tt} = \phi_t+\phi_u u_t+\phi_{u_x}u_{xt}+\phi_{u_{xx}}u_{xxt},$$

the equations (5.332) may be written as

$$T_m^n = k(\tfrac{1}{2}-\gamma_1)\phi_{u_t}u_{tt}+0(k^2+h^2) \quad (5.333)$$

Thus, the values $\gamma_1 = 0$ and 1 give the difference schemes of $0\,(k+h^2)$ and the value $\gamma_1 = 1/2$ gives an high accuracy scheme of $0\,(k^2+h^2)$.

For examining the convergence of (5.329), we determine the error equation. Substituting $\epsilon_m^n = u_m^n - U_m^n$ into (5.329), subtracting (5.328) from (5.329) and applying the mean value theorem, we find

$$[(1-\gamma_1)\epsilon_m^n+\gamma_1\epsilon_m^{n+1}]\phi_u+\left[\frac{\epsilon_m^{n+1}-\epsilon_m^n}{k}\right]\phi_{u_t}+\left[\frac{1}{h}\,\mu_x\delta_x((1-\gamma_1)\epsilon_m^n +\gamma_1\epsilon_m^{n+1})\right]\phi_{u_x}+\left[\frac{1}{h^2}\,\delta_x^2((1-\gamma_1)\epsilon_m^n+\gamma_1\epsilon_m^{n+1})\right]\phi_{u_{xx}} +0\,[k(\tfrac{1}{2}-\gamma_1)+k^2+h^2]=0 \tag{5.334}$$

where the function has been evaluated at an appropriate intermediate point. Simplifying (5.334), the error equation is obtained as

$$\gamma_1\left[-\frac{1}{2h}\phi_{u_x}+\frac{1}{h^2}\phi_{u_{xx}}\right]\epsilon_{m-1}^{n+1}+\left[\frac{1}{k}\,\phi_{u_t}+\gamma_1\left(\phi_u-\frac{2}{h^2}\phi_{u_{xx}}\right)\right]\epsilon_m^{n+1} +\gamma_1\left[\frac{1}{2h}\phi_{u_x}+\frac{1}{h^2}\phi_{u_{xx}}\right]\epsilon_{m+1}^{n+1}=(1-\gamma_1)\left[\frac{1}{2h}\phi_{u_x}-\frac{1}{h^2}\phi_{u_{xx}}\right]\epsilon_{m-1}^n +\left[\frac{1}{k}\phi_{u_t}+(1-\gamma_1)\left(-\phi_u+\frac{2}{h^2}\phi_{u_{xx}}\right)\right]\epsilon_m^n-(1-\gamma_1)\left[\frac{1}{2h}\phi_{u_x}+\frac{1}{h^2}\phi_{u_{xx}}\right] +0[k(\tfrac{1}{2}-\gamma_1)+k^2+h^2] \tag{5.335}$$

This equation enables us to examine the stability and convergence of the difference scheme (5.329). The value $\gamma_1 = 0$ in (5.335) gives an error equation which on simplification can be written as

$$\phi_{u_t}\epsilon_m^{n+1}=r\left[\frac{1}{2}\;h\phi_{u_x}-\phi_{u_{xx}}\right]\epsilon_{m-1}^n+\left[\phi_{u_t}-k\phi_u+2r\phi_{u_{xx}}\right]\epsilon_m^n +r\left[-\frac{1}{2}\,h\phi_{u_x}-\phi_{u_{xx}}\right]+0\,(k^2+kh^2) \tag{5.336}$$

We may now use the maximum analysis to establish the convergence of the difference scheme (5.329) with $\gamma_1 = 0$. We assume that

$$\phi_{u_t}\geqslant a^*>0,\ \phi_{u_{xx}}\leqslant b^*<0$$
$$|\,\phi_u\,|+|\,\phi_{u_x}\,|-\phi_{u_{xx}}\leqslant c^*, \tag{5.337}$$

From (5.336), we get

$$\pm\frac{1}{2}h\phi_{u_x}-\phi_{u_{xx}}\geqslant-\frac{1}{2}hc^*-b^*\geqslant 0$$

or

$$h\leqslant-\frac{2b^*}{c^*}, \tag{5.338}$$

and

$$\phi_{u_t}-k\phi_u+2r\phi_{u_{xx}}\geqslant a^*-kc^*+2rb^*\geqslant 0$$

or

$$0<k\leqslant\frac{h^2a^*}{c^*h^2-2b^*} \tag{5.339}$$

Thus, the convergence and stability of (5.329) with $\gamma_1 = 0$ are assured provided h and k satisfy (5.338) and (5.339) respectively.

The explicit difference scheme ($\gamma_1 = 0$) will require the solution of only linear algebraic equations at each time level. The implicit difference schemes ($\gamma_1 = 1, 1/2$) lead to the nonlinear equations at each time level which must be solved by an iterative process. We now derive implicit difference schemes for a few special cases of (5.328). The difference schemes will be modified such that the resulting algebraic equations are linear.

Let us assume that (5.328) has the form

$$c(x, t, u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(p(x, t)\frac{\partial u}{\partial x}\right) - q(x, t, u) \tag{5.340}$$

The integral identity with

$$K(u; x, t) = c(x, t, u)\frac{\partial u}{\partial t} + q(x, t, u)$$

may be used to obtain the difference schemes for (5.340).

We have

$$\begin{aligned}[(1-\gamma_1)\, c(u_m^{n+1}) + \gamma_1 c(u_m^n)]\, \Delta_t u_m^n - r[(1-\gamma_1)\delta_x(p_m^{n+1}\delta_x u_m^{n+1})\\ + \gamma_1\delta_x(p_m^n\delta_x u_m^n)] + k[(1-\gamma_1)q(u_m^{n+1}) + \gamma_1 q(u_m^n)] = 0\end{aligned} \tag{5.341}$$

where the explicit dependence of c and q on x and t has not been written. The value $\gamma_1 = 1$ gives an explicit scheme

$$c(u_m^n)\Delta_t u_m^n - r\delta_x(p_m^n\delta_x u_m^n) + kq(u_m^n) = 0 \tag{5.342}$$

of order of accuracy $(k+h^2)$.

For $\gamma_1 = 0$, we get the difference scheme

$$c(u_m^{n+1})\Delta_t u_m^n - r\delta_x(p_m^{n+1}\delta_x u_m^{n+1}) + kq(u_m^{n+1}) = 0 \tag{5.343}$$

with error of $O(k+h^2)$.

For $\gamma_1 = 1/2$, the *Crank-Nicolson* scheme is written as

$$\begin{aligned}[c(u_m^{n+1}) + c(u_m^n)]\Delta_t u_m^n - r[\delta_x(p_m^{n+1}\delta_x u_m^{n+1})\\ + \delta_x(p_m^n\delta_x u_m^n)] + k[q(u_m^{n+1}) + q(u_m^n)] = 0\end{aligned} \tag{5.344}$$

with error of $O(k^2+h^2)$.

The implicit schemes (5.343) and (5.344) lead to the nonlinear algebraic equations which must be solved by an iterative process. These equations are nonlinear because $q(x, t, u)$ and $c(x, t, u)$ are to be evaluated at the advance level t_{n+1}. Since $u_m^{n+1} = u_m^n + O(k)$, we modify slightly the difference scheme (5.343); both c and q are evaluated at the known level t_n so that the resulting algebraic equations at each time step remain linear and can be solved easily by elimination. The modified *Douglas* difference scheme is given by

$$c(x_m, t_{n+1}, u_m^n)\Delta_t u_m^n - r\delta_x(p_m^{n+1}\delta_x u_m^n) + kq(x_m, t_{n+1}, u_m^n) = 0 \tag{5.345}$$

which still retains the order of accuracy $(k+h^2)$. The nonlinear algebraic equations in (5.344) arise from the use of $[q(x_m, t_{n+1}, u_m^{n+1})+q(x_m, t_n, u_m^n)]$ and a similar relation for c. We approximate it by $2q(x_m, t_{n+1/2}, u_m^{n+1/2})$ and determine $u_m^{n+1/2}$ in terms of u_m^n. Using the differential equation, such an approximation is easily obtained as

$$\begin{aligned} u_m^{n+1/2} &= u_m^n+\frac{1}{2}\,k\,\frac{\partial u_m^n}{\partial t}+0(k^2) \\ &= u_m^n+\frac{1}{2}\,k\left\{\left[\frac{\partial}{\partial x}\left(p\,\frac{\partial u}{\partial x}\right)-q(x,t,u)\right]\Big/c(x,t,u)\right\}_m^n+0(k^2) \\ &= u_m^n+\frac{k}{2c(u_m^n)}\,[h^{-2}\delta_x(p_m^n\delta_x u_m^n)-q(u_m^n)] \end{aligned} \tag{5.346}$$

Equation (5.344) with the help of (5.346) can be written as

$$\begin{aligned} 2c(u_m^{n+1/2})(u_m^{n+1}-u_m^n) &= r[\delta_x(p_m^{n+1/2}\delta_x u_m^{n+1}) \\ &\quad +\delta_x(p_m^{n+1/2}\delta_x u_m^n)]+2kq(u_m^{n+1/2}) \end{aligned} \tag{5.347}$$

where

$$u_m^{n+1/2} = u_m^n+\frac{k}{2c(u_m^n)}\,[h^{-2}\delta_x(p_m^{n+1/2}\delta_x u_m^n)-q(u_m^n)]$$

The evaluation of p at $t = t_{n+1/2}$ is done to simplify the calculations in implementation of (5.347). It does not affect the order of accuracy of the difference scheme. Equations (5.347) are linear algebraic equations and can be solved by elimination. For $q(x, t, u) \equiv 0$ and $p(x, t) = 1$, the differential equation (5.340) becomes

$$c(x,t,u)\,\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \tag{5.348}$$

and for which the difference scheme of $0(k^2+h^4)$ is found to be

$$\begin{aligned} &[c(u_m^{n+1})+c(u_m^n)]\left(\frac{u_m^{n+1}-u_m^n}{r}\right) \\ &= \delta_x^2\left[1-\frac{1}{6r}\,c_m^n\right]u_m^{n+1}+\delta_x^2\left[1+\frac{1}{6r}\,c_m^n\right]u_m^n \end{aligned} \tag{5.349}$$

Notice that the algebraic equations in (5.349) are nonlinear as a result of the coefficient of the time difference. In general, they must be solved by iteration. This can be accomplished by predicting an approximate value $u_m^{[0]n+1}$ by some means such as

$$u_m^{[0]n+1} = u_m^n+r\,\frac{\delta_x^2 u_m^n}{c(u_m^n)}$$

or

$$u_m^{[0]n+1} = 2u_m^n-u_m^{n-1}$$

using $u_m^{[0]n+1}$ to evaluate the difference coefficient, and solving the linear equations for $u_m^{[1]n+1}$ by elimination. Then, find $u_m^{[2]n+1}$ from $u_m^{[1]n+1}$, etc. Unless $\partial c/\partial u$ is quite large, the resulting iteration should converge quite rapidly. Next, consider that Equation (5.328) is of the form

$$c\left(u, \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f\left(u, \frac{\partial u}{\partial x}\right) \tag{5.350}$$

The *Lees* difference scheme for (5.350) is based on the linearized *Crank-Nicolson* difference scheme and it is given by

$$c\left(u_m^{n+1/2}, \frac{1}{h}\mu_x\delta_x u_m^{n+1/2}\right)(u_m^{n+1} - u_m^n) = \frac{r}{2}\delta_x^2(u_m^{n+1} + u_m^n) + kf\left(u_m^{n+1/2}, \frac{1}{h}\mu_x\delta_x u_m^{n+1/2}\right) \tag{5.351}$$

where

$$u_m^{n+3/2} = \frac{3}{2}u_m^{n+1} - \frac{1}{2}u_m^n$$

and

$$u_m^{1/2} = u_m^{(0)} + \frac{k}{2c\left(u_m^0, \frac{1}{h}\mu_x\delta_x u_m^0\right)}\left[h^{-2}\delta_x^2 u_m^0 + f\left(u_m^0, \frac{1}{h}\mu_x\delta_x u_m^0\right)\right] \tag{5.352}$$

If f is a linear function of u, we may replace

$$f\left(u_m^{n+1/2}, \frac{1}{h}\mu_x\delta_x u_m^{n+1/2}\right) \text{ by } f\left((u_m^{n+1} + u_m^n)/2, \frac{1}{h}\mu_x\delta_x u_m^{n+1/2}\right),$$

and if f is a linear function of $(\partial u/\partial x)$, we may replace

$$f\left(u_m^{n+1/2}, \frac{1}{h}\mu_x\delta_x u_m^{n+1/2}\right) \text{ by } f\left(u_m^{n+1/2}, \frac{1}{h}\mu_x\delta_x(u_m^{n+1} + u_m^n)\right)$$

The above scheme is also termed as *extrapolated Crank-Nicolson* difference scheme.

The *Douglas-Jone* scheme or predictor-corrector method for (5.350) can be written as

$$P: c\left(u_m^n, \frac{1}{2h}\delta_x u_m^n\right)(u_m^{n+1/2} - u_m^n) = \frac{r}{2}\delta_x^2 u_m^{n+1/2} + \frac{k}{2}f\left(u_m^n, \frac{1}{2h}\delta_x u_m^n\right)$$

$$C: c\left(u_m^{n+1/2}, \frac{1}{2h}\delta_x u_m^{n+1/2}\right)(u_m^{n+1} - u_m^n) = \frac{r}{2}\delta_x^2(u_m^{n+1} + u_m^n) + kf\left(u_m^{n+1/2}, \frac{1}{2h}\delta_x u_m^{n+1/2}\right) \tag{5.353}$$

It is clear that the *P-C* methods give rise to linear algebraic equations to be solved at each time step.

where

$$c = \frac{k}{h} \text{ and } r = \frac{k}{h^2}.$$

For linearization we may put

$$u_m^{n+1} = u_m^n + v_m^n$$

where $$v_m^n = u_m^{n+1} - u_m^n,$$

into the above system of nonlinear equations and neglect the term $0((v_m^n)^2)$ to get the following system of equations

$$v_m^n + \frac{1}{2} c\mu_x\delta_x(u_m^n v_m^n) - \frac{1}{2} r\delta_x^2 v_m^n$$

$$= r\delta_x^2 u_m^n - \frac{1}{2} c\mu_x\delta_x(u_m^n)^2$$

or $$-\left(\frac{1}{2}r + \frac{1}{4} cu_{m-1}^n\right)v_{m-1}^n + (1+r)v_m^n - \left(\frac{1}{2} r - \frac{1}{4} cu_{m+1}^n\right)v_{m+1}^n$$

$$= \left(ru_{m-1}^n + \frac{1}{4}c(u_{m-1}^n)^2\right) - 2ru_m^n + \left(ru_{m+1}^n - \frac{1}{4}c(u_{m+1}^n)^2\right)$$

Substituting $r = \frac{1}{6}$, $c = \frac{1}{18}$ and simplifying, we obtain

$$-(6+u_{m-1}^n)v_{m-1}^n + 84v_m^n - (6-u_{m+1}^n)v_{m+1}^n$$

$$= (12+u_{m-1}^n)\, u_{m-1}^n - 24u_m^n + (12-u_{m+1}^n)\, u_{m+1}^n, n = 0, 1, 2, \ldots$$

$$1 \leqslant m \leqslant 2$$

The initial and boundary conditions become

$$u_1^0 = \frac{2\pi\sqrt{3}}{5} = 2.1766,\ u_2^0 = \frac{2\pi\sqrt{3}}{3} = 3.6276$$

$$u_0^n = 0,\ u_3^n = 0,\ n = 0, 1, 2, \ldots$$

We have:

$$n = 0, -(6+u_{m-1}^0)\, v_{m-1}^0 + 84v_m^0 - (6-u_{m+1}^0)\, v_{m+1}^0$$

$$= (12+u_{m-1}^0)\, u_{m-1}^0 - 24u_m^0 + (12-u_{m+1}^0)\, u_{m+1}^0$$

$$m = 1, -(6+u_0^0)\, v_0^0 + 84v_1^0 - (6-u_2^0)\, v_2^0$$

$$= (12+u_0^0)\, u_0^0 - 24u_1^0 + (12-u_2^0)\, u_2^0$$

$$m = 2, -(6+u_1^0)\, v_1^0 + 84v_2^0 - (6-u_3^0)\, v_3^0$$

$$= (12+u_1^0)\, u_1^0 - 24u_2^0 + (12-u_3^0)\, u_3^0$$

or
$$84v_1^0 - 2.3724v_2^0 = -21.8667$$
$$-8.1766v_1^0 + 84v_2^0 = -56.2056$$
$$v_1^0 = -0.2799 \qquad v_2^0 = -0.6933$$
$$u_1^1 = u_1^0 + v_1^0 = 2.1766 - 0.2799 = 1.8967$$
$$u_2^1 = u_2^0 + v_2^0 = 3.6276 - 0.6963 = 2.9313$$

The analytic solution is given by

$$u(x, t) = \frac{2\pi e^{-\pi^2 t} \sin \pi x}{2 + e^{-\pi^2 t} \cos \pi x}$$

We obtain the solution values

$$u(x_1, t_1) = 1.8756$$
$$u(x_2, t_1) = 2.8623$$

5.12 DIFFERENCE SCHEMES FOR EQUATIONS WITH CYLINDRICAL SYMMETRY

The cylindrical symmetric diffusion equation is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \tag{5.361}$$

where r is the radial space variable. The appropriate initial, boundary and regularity conditions are of the form

$$u(r, 0) = f(r), \frac{\partial u(0, t)}{\partial r} = 0, u(R, t) = g(t) \tag{5.362}$$

where $0 \leqslant r \leqslant R$. We define the mesh points in the r-t plane by the intersection of the circles $r_m = mh$, $m = 1, 2, \ldots$ and the lines $t = nk$, $k = 0, 1, 2, \ldots$ and denote the approximate values of u at these points by u_m^n, h and k being the mesh spacings in the space and time directions respectively.

A simple explicit difference scheme to (5.361), of $0\,(k+h^2)$, is written as

$$u_m^{n+1} = \lambda\left(1 - \frac{p}{2}\right)u_{m-1}^n + (1-2\lambda)u_m^n + \lambda\left(1 + \frac{p}{2}\right)u_{m+1}^n \tag{5.363}$$

where $\lambda = k/h^2$ and $p = h/r_m$.

The stability analysis derived here is as in the case of a constant coefficient difference scheme, the variable coefficient $r_m = mh$ is taken care by studying the stability limit for $m = 1, 2, \ldots$. Applying the von Neumann method to (5.363), we obtain the stability condition $\lambda < \frac{1}{2}$.

$$= \frac{N}{N+h(\eta)}$$

where N represents numerator in (5.376) and

$$h(\eta) = 4(1-\eta)f(\eta), \qquad \eta = \cos \beta h$$

$$f(\eta) = (b_1+b_2)\left[(b_1-b_2)(1-\eta)+\frac{r_m}{3}(\eta+2)\right]$$

$$+(1+\eta)(c_1+c_2)\left(c_1-c_2-\frac{h}{3}\right).$$

For stability, we require $|\xi| \leqslant 1$. Hence, if $h(\eta) > 0$. the difference scheme (5.374) is stable. Since $|\eta| \leqslant 1$, this gives us the condition that $f(\eta) \geqslant 0$. We have

$$b_1+b_2 > 0, \qquad b_1-b_2 > 0$$

$$f(-1) = (b_1+b_2)\left[2(b_1-b_2)+\frac{r_m}{3}\right] > 0$$

$$f(1) = \frac{\lambda h^2}{(12+5/m^2)^2}\left[4m\left(12+\frac{5}{m^2}\right)\left(3m+\frac{1}{m}\right)\right.$$

$$\left.-\left(2+\frac{1}{m^2}\right)\left(6+\frac{4}{m^2}+\frac{3\lambda}{m^2}\right)\right], \; m = 1, 2, \ldots$$

where we have substituted $r_m = mh$.

For $m = 1, f(1) > 0$ if $\lambda < \frac{242}{9}$ while for $m \geqslant 2, f(1) > 0$ is satisfied for $\lambda < \frac{242}{9}$. Hence, the method (5.374) is stable for $\lambda < \frac{242}{9}$.

5.12.2 Approximation at the boundary

Consider the differential equation at the node $r = 0$. We need to write an approximation for $\frac{\partial u}{\partial r}$ at $r = 0$. We have

$$\lim_{r\to 0}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) = \left(\frac{\partial^2 u}{\partial r^2}\right)_{r\to 0} + \lim_{r\to 0}\frac{\partial u/\partial r}{r}$$

$$= 2\frac{\partial^2 u}{\partial r^2}$$

Thus, the differential equation (5.361) for $r \to 0$ becomes

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial r^2} \tag{5.377}$$

The difference schemes for (5.377) can easily be obtained, using the relation $u_{-1}^n = u_1^n$. Thus, we have the following difference schemes of

(i) $0(k+h^2)$

$$u_0^{n+1} = (1-4\lambda)u_0^n+4\lambda u_1^n \tag{5.378}$$

(ii) $0(k^2+h^2)$

$$[1+4\lambda(1-\gamma_1)]u_0^{n+1} - 4\lambda(1-\gamma_1)u_1^{n+1} = (1-4\lambda\gamma_1)u_0^n + 4\lambda\gamma_1 u_1^n \quad (5.379)$$

(iii) $0(k^2+h^4)$

$$(5+12\lambda)u_0^{n+1} + (1-12\lambda)u_1^{n+1} = (5-12\lambda)u_0^n + (1+12\lambda)u_1^n \quad (5.380)$$

5.12.3 Two space variables

Here we consider the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \quad (5.381)$$

subject to the initial and boundary conditions of the form (5.153) and the condition

$$\frac{\partial u(0, z, t)}{\partial r} = 0 \quad (5.382)$$

The explicit and implicit difference schemes discussed in Section 5.12.2 can easily be extended in the case of the two dimensional equation (5.381). For example, the explicit scheme (5.363) becomes

$$u_{l,m}^{n+1} = \lambda\left(1-\frac{p}{2}\right)u_{l-1,m}^n + (1-4\lambda)u_{l,m}^n + \lambda\left(1+\frac{p}{2}\right)u_{l+1,m}^n + \lambda(u_{l,m+1}^n + u_{l,m-1}^n) \quad (5.383)$$

where $u_{l,m}^n$ is approximate value of $u(lh, mh, nk)$,

$$r_l = lh,\ z_m = mh,\ t_n = nk \text{ and } p = \frac{h}{r_l}.$$

The stability condition for (5.383) is $\lambda \leqslant \frac{1}{4}$. For $r = 0$, the differential equation (5.381) becomes

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} \quad (5.384)$$

To use with the explicit scheme (5.383) we write an $0(k+h^2)$ approximation to (5.384) as

$$u_{0,m}^{n+1} = (1-6\lambda)u_{0,m}^n + 4\lambda u_{1,m}^n + \lambda(u_{0,m+1}^n + u_{0,m-1}^n) \quad (5.385)$$

where we have $u_{-1,m}^n = u_{1,m}^n$. The equation (5.385) is stable for $\lambda \leqslant \frac{1}{6}$.

The Peaceman-Rachford ADI method for (5.381) and (5.384) may be written as

(i) $$\frac{u_{l,m}^{n+1/2} - u_{l,m}^n}{k/2} = h^{-2}\delta_r^2 u_{l,m}^{n+1/2} + \frac{1}{r_l}\mu_r\delta_r u_{l,m}^{n+1/2} + h^{-2}\delta_z^2 u_{l,m}^n$$

Solving we get

$$u_0^1 = 0.8576\ u_1^1 = 0.7270\ u_2^1 = 0.3903$$

5.12.4 Results from computation

We have solved the differential equation (5.361) subject to the following initial and boundary conditions:

(i) $u(r, 0) = J_0(\alpha r), \quad 0 \leqslant r \leqslant 1$

$$\frac{\partial u(0, t)}{\partial r} = 0, \quad u(1, t) = 0$$

where α is the first root of $J_o(\alpha) = 0$, with the exact solution

$u(r, t) = J_o(\alpha r) \exp[-\alpha^2 t]$

(ii) $u(r, 0) = J_o(r), \quad 2 \leqslant r \leqslant 3$

$$u(2, t) = J_o(2) \exp(-t),\ u(3, t) = J_o(3) \exp(-t) \tag{5.388}$$

with the exact solution

$$u(r, t) = J_o(r) \exp(-t)$$

Using the schemes (5.365), $\gamma_1 = \frac{1}{2}$ and (5.374), we have done the integration with $h = .1$. The integration is carried upto $t = 4.5$ with values of $\lambda = 0.3, 0.5, 1, 3, 5$. The maximum absolute errors are given in Table 5.13. It is noted that the error in the numerical solution increased considerably in the beginning for both implicit methods and then decreased. The computations show that the method of $0(k^2+h^4)$ produces better results than the method of $0(k^2+h^2)$.

TABLE 5.13 Maximum Absolute Error, $h = 0.1$

λ	Steps	*Problem (i)* Formula (5.365)	*Problem (i)* Formula (5.374)	*Problem (ii)* Formula (5.365)	*Problem (ii)* Formula (5.374)
0.3	500	0.261−4	0.980−8	0.127—5	0.199—8
	1000	0.916−8	0.348—11	0.282−6	0.446−9
	1250	0.154—9	0.000	0.133−6	0.211—9
0.5	300	0.247−4	0.810−7	0.127—5	0.298—8
	600	0.891−8	0.279—10	0.283−6	0 666—9
	750	0.150−9	0.000	0.155−6	0.315—9
1.0	150	0209—4	0.405—6	0.127—5	0.867—8
	300	0.819−8	0.139—9	0.283−6	0.193—8
	375	0.140—9	0.000	0.137—6	0.913—9
3.0	50	0.306—5	0.575−5	0.130—5	0.256−6
	100	0.403−8	0.127—8	0.290—6	0.278—7
	125	0.805−10	0.206—10	0.137—6	0.131—7
5.0	30	0.308−4	0.103—4	0.136—5	0 371—6
	60	0.850—8	0.347—8	0.303—6	0.828—7
	75	0.144—9	0.534−10	0.143−6	0.391—7

Bibliographical Note

An excellent survey of numerical methods for the parabolic equations with an extensive bibliography is given in 69. The other texts that deal with the difference methods for parabolic equations are 9, 12, 86, 184, 203, 217 and 256. Particularly 184 and 217 are recommended.

The difference schemes for the heat flow equations in one space dimension are available in the following references; explicit schemes, 72, 107, 218, implicit schemes, 53, 56, 131, 138, 155 and 160. The von Neumann method has been used for the stability of the difference schemes in 189. The Crank-Nicolson difference scheme has been applied to solve numerically the parabolic equation with mixed boundary conditions in 29 and the stability analysis has been carried out with the help of the matrix method in 236. An explicit unconditionally stable scheme for the heat flow equation in two space variable is given in 164.

The alternating direction implicit (ADI) methods have been discussed in 15, 17, 31, 71, 73, 78, 79, 97, 124 and 195. The difference schemes for the parabolic equations with variable coefficients and with or without mixed derivatives are found in 123, 124, 182 and 233. The stability analysis of the difference scheme is examined by using the Widlund analysis in 254. The difference schemes for the solution of the fourth order parabolic equations are given in 6, 45, 47, 52, 75, 77 and 132.

The nonlinear parabolic equations are studied in 70 and 170. The convection-diffusion equation is discussed in 224. The cylindrical heat conduction equation is treated in 7 and 125.

Problems

1. The function $u(x, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}+cu,$$

where c is a constant, and possesses continuous and finite derivatives of sufficiently high order. Obtain the discretization error of the formulas

(i) $$\left[1-\frac{1}{2}crh^2-\frac{1}{2}r\delta_x^2\right]u_m^{n+1} = \left[1+\frac{1}{2}crh^2+\frac{1}{2}r\delta_x^2\right]u_m^n$$

(ii) $$\left[1-\frac{1}{2}crh^2-\frac{1}{2}\left(r-\frac{1}{6}\left(1-\frac{1}{2}crh^2\right)\right)\delta_x^2\right]u_m^{n+1}$$
$$=\left[1+\frac{1}{2}crh^2+\frac{1}{2}\left(r+\frac{1}{6}\left(1+\frac{1}{2}crh^2\right)\right)\delta_x^2\right]u_m^n$$

What stability criterion is applicable?

where $h = 1/M$ when applied to solve the differential equation $\partial u/\partial t = \partial^2 u/\partial x^2$ in the domain $\mathcal{R} = [0 \leqslant x \leqslant 1]\times[t \geqslant 0]$ subject to the initial and the first boundary conditions lead to the system of $(M-1)$ linear equations in $(M-1)$ unknowns of the form

$$\mathbf{A}\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n+\mathbf{f}^n$$

where

$$\mathbf{u}^s = [u_1^s \; u_2^s \; \ldots \; u_{M-1}^s]^T, \; s = n, n+1.$$

Find

(i) the forms of A and **B**;
(ii) the stability condition for all $\alpha > 0$;
(iii) the truncation error at the point $(mh, (n+1)k)$

The system of equations is solved with the help of the Jacobi iterative method and the Gauss-Seidel iterative method respectively. What convergence criterion is applicable? Define the SOR method and obtain the optimum relaxation factor.

9. Consider the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

in the region $\mathcal{R} = [0 \leqslant x \leqslant 1]\times[t \geqslant 0]$, subject to the following initial and mixed boundary conditions

$$u = f(x), \; t = 0, \; 0 \leqslant x \leqslant 1$$

$$\frac{\partial u}{\partial x} = 0, \; x = 0, \; t > 0$$

$$\frac{\partial u}{\partial x} = -cu+d, \; x = 1, \; t > 0$$

where c and d are positive constants. Derive a matrix equation of the form

$$\mathbf{u}^{n+1} = \mathbf{A}\mathbf{u}^n+\mathbf{b}$$

for an explicit difference scheme

$$u_m^{n+1} = (1-2r)u_m^n+r(u_{m-1}^n+u_{m+1}^n)$$

where $r = k/h^2$ and $h = 1/M$.

Use central differences to approximate the boundary conditions. Show that this difference scheme is stable when $r \leqslant 1/(2+ch)$.
Derive also a matrix equation of the form

$$\mathbf{B}\mathbf{u}^{n+1} = (4\mathbf{I}-\mathbf{B})\mathbf{u}^n+\mathbf{f}$$

for this partial differential equation by using a Crank-Nicolson implicit difference scheme and show that this is always stable. Use central differences to approximate the boundary conditions. Discuss briefly the stability of the two methods when c is a negative constant.

10. The difference representation

$$u_m^{n+1} = r(1-ah)u_{m-1}^n + \left(1-2r\left(1-\frac{1}{2}bh^2\right)\right)u_m^n + r(1+ah)u_{m+1}^n,$$

$$1 \leqslant m \leqslant M-1, n = 0, 1, 2, \ldots$$

where $r = k/h^2$, is to be used for the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2a\frac{\partial u}{\partial x} + bu$$

subject to the initial and first boundary conditions over the region

$$\mathfrak{R} = [0 \leqslant x \leqslant 1] \times [t \geqslant 0]$$

Show that this would give rise to a recurrence relation

$$\mathbf{u}^{n+1} = \mathbf{A}\mathbf{u}^n + \mathbf{c}$$

where

$$\mathbf{u}^s = [u_1^s\, u_2^s \ldots u_m^s]^T,\ s = n, n+1$$

giving the form of $\mathbf{A}$. Derive the stability conditions for this scheme.

11. Prove that when the difference scheme

$$u_m^{n+1} = ru_{m-1}^n + (1-2r)u_m^n + ru_{m+1}^n$$

is used to solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad -\infty < x < \infty, t = 0$$

with the initial condition

$$u(x, 0) = f(x), -\infty < x < \infty, t > 0$$

and it is assumed that u has bounded derivatives of sufficiently high order then

$$|\epsilon_m^n| \leqslant \frac{1}{2}\left(r + \frac{1}{6}\right)h^2 M_4 t_n$$

where

$$0 < r \leqslant \frac{1}{2}, \left|\frac{\partial^4 u}{\partial x^4}\right| \leqslant \max_{-\infty<x<\infty} |f^{(4)}(x)| = M_4$$

and the round-off error is neglected.

Show also that

$$|\epsilon_m^n| \leqslant \frac{1}{135}h^4 M_6 t_n$$

when $r = \frac{1}{6}$, $M_6 = \max_{-\infty<x<\infty} |f^{(6)}(x)|$

Use the von Neumann method of stability analysis to show that formula (i) is stable for $0 < r \leqslant 1/(4-ch^2/2)$, $c \lessdot 0$ but formula (ii) is unconditionally stable for $c \leqslant 0$. Also, formula (i) is relatively stable for

$$0 < r \leqslant \frac{1}{4-\frac{1}{2}ch^2}, h < \sqrt{8/c}, c > 0$$

18. Find the order of accuracy of the difference scheme

$$\left[1-\frac{r}{2\left(1-\frac{r}{2}ch^2\right)}\delta_x^2\right]\left[1-\frac{r}{2\left(1-\frac{r}{2}ch^2\right)}\delta_y^2\right]u_{l,m}^{n+1}$$

$$=\frac{1+\frac{r}{2}ch^2}{1-\frac{r}{2}ch^2}\left[1+\frac{r}{2\left(1+\frac{r}{2}ch^2\right)}\delta_x^2\right]\left[1+\frac{r}{2\left(1+\frac{r}{2}ch^2\right)}\delta_y^2\right]u_{l,m}^{n}$$

where $u(x, y, t)$ satisfies the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+cu$$

19. *The* Peaceman-Rachford ADI method for the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+cu$$

may be written in the form

$$\frac{u_{l,m}^{n+1/2}-u_{l,m}^{n}}{\frac{k}{2}} = h^{-2}\delta_x^2 u_{l,m}^{n+1/2}+h^{-2}\delta_y^2 u_{l,m}^{n}+cu_{l,m}^{n+1/2}$$

$$\frac{u_{l,m}^{n+1/2}-u_{l,m}^{n+1}}{\frac{k}{2}} = h^{-2}\delta_x^2 u_{l,m}^{n+1/2}+h^{-2}\delta_y^2 u_{l,m}^{n+1}+cu_{l,m}^{n+1}$$

Derive an equation relating $u_{l,m}^{n+1}$ and $u_{l,m}^{n}$ and show that the resulting difference scheme is both consistent and unconditionally stable.

20. The Douglas-Rachford method for solving

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

is given by

$$(1-r\delta_x^2)u_{l,m}^{*n+1} = (1+r\delta_y^2)u_{l,m}^{n}$$

$$(1-r\delta_y^2)u_{l,m}^{n+1} = u_{l,m}^{*n+1}-r\delta_y^2 u_{l,m}^{n}$$

Show that the truncation error of the formula is of $O(k+h^2)$. Write the difference scheme in the D'Yakonov split form.

21. Consider the heat flow equation in two space dimensions

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

in the domain $\mathcal{R}=[0 \leqslant x, y \leqslant 1]\times[t \geqslant 0]$, subject to the initial and boundary conditions

$$u(0, x, y) = \sin \pi\alpha x \sin \pi\beta y,\ 0 \leqslant x, y \leqslant 1$$
$$u(t, x, y) = 0,\ t > 0$$

on the boundary of the domain.
The theoretical solution of the problem is

$$u(x, y, t) = \exp\,[-\pi^2(\alpha^2+\beta^2)t] \sin \alpha\pi x \sin \beta\pi y$$

Obtain the decay factor of the following difference schemes:

(i) $\nabla_t u_{l,m}^{n+1} = r(\delta_x^2+\delta_y^2)u_{l,m}^{n}$

(ii) $\nabla_t u_{l,m}^{n+1} = r(\delta_x^2+\delta_y^2)u_{l,m}^{n+1}$

(iii) $\left[1-\frac{1}{2}r\delta_x^2\right]\left[1-\frac{1}{2}r\delta_y^2\right]u_{l,m}^{n+1} = \left[1+\frac{1}{2}r\delta_x^2\right]\left[1+\frac{1}{2}r\delta_y^2\right]u_{l,m}^{n}$

(iv) $\left[1-\frac{1}{2}\left(r-\frac{1}{6}\right)\delta_x^2\right]\left[1-\frac{1}{2}\left(r-\frac{1}{6}\right)\delta_y^2\right]u_{l,m}^{n+1}$

$$=\left[1+\frac{1}{2}\left(r+\frac{1}{6}\right)\delta_x^2\right]\left[1+\frac{1}{2}\left(r+\frac{1}{6}\right)\delta_y^2\right]u_{l,m}^{n}$$

Compare, in the limiting case $h \to 0$, the decay factor of these various methods and state which one of the methods will give more accurate results.

22. The two step difference method

$$\frac{u_{l,m}^{n+1/2}-u_{l,m}^{n}}{k} = ah^{-2}\delta_x^2\, u_{l,m}^{n+1/2}+\frac{1}{4}\,bh^{-2}H_xH_y u_{l,m}^{n}$$

$$\frac{u_{l,m}^{n+1}-u_{l,m}^{n+1/2}}{k} = \frac{1}{4}\,bh^{-2}H_xH_y u_{l,m}^{n+1/2}+h^{-2}c\delta_y^2\, u_{l,m}^{n+1}$$

where a, b and c are constants and

$$H_x u_{l,m}^{n} = u_{l+1,m}^{n}-u_{l-1,m}^{n}$$
$$H_y u_{l,m}^{n} = u_{l,m+1}^{n} - u_{l,m-1}^{n}$$

is used to solve the equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}+2b\,\frac{\partial^2 u}{\partial x\partial y}+c\,\frac{\partial^2 u}{\partial y^2}$$

$$a > 0,\ c > 0,\ b^2-ac < 0$$

Find the principal part of the truncation error of the difference scheme which is a relation between $u_{l,m}^{n+1}$ and $u_{l,m}^{n}$.
Also, prove that the difference scheme is unconditionally stable.

Determine the differential equation to which the difference scheme will correspond to according as

(i) $k \to 0,\ h \to 0, \left[\frac{k}{h}\right] \to 0$

(ii) $k \to 0,\ h \to 0, \left[\frac{k}{h}\right] = c$

The Jacobi and Gauss-Seidel iterative methods are used to solve the system, find the convergence conditions.

28. The fourth order parabolic equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0$$

may be replaced by a system of equations

$$\frac{\partial v}{\partial t} = -\frac{\partial^2 w}{\partial x^2},\ \frac{\partial w}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

Consider the following schemes:

(i) $v_m^{n+1} - v_m^n = -r\delta_x^2 w_m^n$

$w_m^{n+1} - w_m^n = r\delta_x^2 v_m^{n+1}$

(ii) $v_m^{n+1} = v_m^{n-1} - 2r(\delta_x^2 w_m^n + b\delta_t^2 w_m^n)$

$w_m^{n+1} = w_m^{n-1} + 2r(\delta_x^2 v_m^n + b\delta_t^2 v_m^n)$

where b is an arbitrary parameter.

(a) Show that (i) is stable if $r < 1/2$.

(b) Show that the truncation error in scheme (ii) is of $O(k^2 + h^2 + (k/h)^2)$ and the scheme is stable for $0 < r^2 < 1/16(b+1)$.

29. The equation

$$\frac{\partial^2 u}{\partial t^2} + a^2\frac{\partial^4 u}{\partial x^4} = b^2\frac{\partial^2 u}{\partial x^2}$$

with appropriate initial and boundary conditions governs the vibration of a bar under tension. The equivalent system of equations is given by

$$\frac{\partial v}{\partial t} = -a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x}$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial v}{\partial x}$$

where $v = \frac{\partial u}{\partial t}$ and $w = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x}$

Consider the following difference schemes:

(i) $$\frac{v_m^{n+1}-v_m^n}{k} = -ah^{-2}\delta_x^2 w_m^n + bh^{-1}\mu_x\delta_x w_m^n$$

$$\frac{w_m^{n+1}-w_m^n}{k} = ah^{-2}\delta_x^2 v_m^{n+1} + bh^{-1}\mu_x\delta_x v_m^{n+1}$$

(ii) $$\frac{v_m^{n+1}-v_m^n}{k} = -\tfrac{1}{2}ah^{-2}(\delta_x^2 w_m^{n+1} + \delta_x^2 w_m^n) + \tfrac{1}{4}bh^{-1}(w_{m+1}^{n+1} - w_{m-1}^{n+1} + w_{m+1}^n - w_{m-1}^n)$$

$$\frac{w_m^{n+1}-w_m^n}{k} = \tfrac{1}{2}ah^{-2}(\delta_x^2 v_m^{n+1} + \delta_x^2 v_m^n) + \tfrac{1}{4}bh^{-1}(v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n)$$

Show that formula (i) is stable for $0 < ar < 1/2$ and formula (ii) is stable for all values of r.

30. Obtain a difference approximation to the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[u\frac{\partial u}{\partial x}\right]$$

with initial and boundary conditions

$$u(x, 0) = 300 + 3(x-4)^2,\ 0 \leqslant x \leqslant 4$$
$$u(0, t) = 348,\ u(4, t) = 300,\ t \geqslant 0$$

Use the formula valid for $\rho(x) \in C^2$, $u(x) \in C^4$

$$\frac{\partial}{\partial x}\left[\rho(x)\frac{\partial u}{\partial x}\right] = \frac{1}{2h^2}((\rho_{m+1}+\rho_m)(u_{m+1}-u_m) - (\rho_m+\rho_{m-1})(u_m-u_{m-1})) + 0(h^2)$$

where

$$u(x_0+mh) = u_m, \text{ etc.}$$

Choose $x_0 = 0$, $h = 1$, $k = 0.001$ and integrate until $t = 0.003$.

(BIT 4(1964), 197)

6

Difference Methods for Hyperbolic Partial Differential Equations

6.1 INTRODUCTION

The hyperbolic type partial differential equations are usually associated with initial value problems or initial boundary value problems. Thus, the derivation of the difference schemes for the hyperbolic equations follows the similar procedure as that of parabolic equations in the previous chapter. We now discuss difference schemes for some important hyperbolic equations in one or more space variables.

6.2 DIFFERENCE SCHEMES FOR HYPERBOLIC EQUATIONS IN ONE SPACE VARIABLE WITH CONSTANT COEFFICIENTS

The simplest hyperbolic problem is that of the vibrating string

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{6.1}$$

in the domain $\mathcal{R} = [0 \leqslant x \leqslant 1] \times [t > 0]$, satisfying the following initial conditions

$$\begin{aligned} u(x, 0) &= f_1(x) \\ \frac{\partial u(x, 0)}{\partial t} &= f_2(x), \quad \text{for} \quad 0 \leqslant x \leqslant 1 \end{aligned} \tag{6.2}$$

nd boundary conditions

$$\begin{aligned} u(0, t) &= g_1(t) \\ u(1, t) &= g_2(t), \quad \text{for all } t > 0 \end{aligned} \tag{6.3}$$

We place a mesh of points (x_m, t_n) on $\mathcal{R}$, where

$$\begin{aligned} x_m &= mh, \; m = 0, 1, 2, \ldots, M, \; Mh = 1, \\ t_n &= nk, \; n = 0, 1, 2, \ldots \end{aligned} \tag{6.4}$$

The exact difference replacement of (6.1) at the nodal point (x_m, t_n) is given by

$$\left(\sinh^{-1}\frac{\delta_t}{2}\right)^2 u(x_m, t_n) = p^2\left(\sinh^{-1}\frac{\delta_x}{2}\right)^2 u(x_m, t_n) \tag{6.5}$$

where $p = k/h$ is the mesh ratio and

$$4\left(\sinh^{-1}\frac{\delta}{2}\right)^2 = \delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 - \dots \tag{6.6}$$

The explicit and implicit difference schemes for (6.1) will be obtained by approximating equation (6.5).

6.2.1 Explicit dfference schemes

An explicit difference scheme for (6.1) is given by

$$\delta_t^2 u_m^n = p^2 \delta_x^2 u_m^n$$

which may be written in the form

$$u_m^{n+1} = 2(1-p^2)u_m^n + p^2(u_{m-1}^n + u_{m+1}^n) - u_m^{n-1} \tag{6.7}$$

where u_m^n is the approximation to $u(x_m, t_n)$. This scheme in schematic form becomes

	-1		
p^2	$2(1-p^2)$	p^2	$u = 0$
	-1		

If each term in (6.7) is expanded in the Taylor series about the nodal point (x_m, t_n) and the function $u(x_m, t_n)$ satisfies (6.1), then we get the truncation error

$$T_m^n = k^2h^2\left[\frac{1}{12}(p^2-1)\frac{\partial^4 u(x_m, t_n)}{\partial x^4} + \frac{1}{360}h^2(p^4-1)\frac{\partial^6 u(x_m, t_n)}{\partial x^6} + \dots\right] \tag{6.8}$$

For $p = 1$, the truncation error vanishes and so the exact difference representation of (6.1) is obtained as

$$u_m^{n+1} = u_{m-1}^n + u_{m+1}^n - u_m^{n-1} \tag{6.9}$$

In order to start computation we require data on the two line $t = 0$ and $t = k$. The first condition in (6.2) gives u_m^0 on the initial lines as

$$u_m^0 = f_1(mh),\ 0 \leqslant m \leqslant M \tag{6.10}$$

We can use the second condition in (6.2) to find values on the line $t = k$. Substituting the central difference approximation for the derivative, i.e.

$$\frac{\partial u_m^0}{\partial t} = \frac{u_m^1 - u_m^{-1}}{2k}$$

in the second condition in (6.2) and eliminating u_m^{-1} from (6.7) for $n = 0$, we get the formula to give the values on the first level

$$u_m^1 = (1-p^2) f_1(mh) + k f_2(mh) + \frac{1}{2} p^2 [f_1(\overline{m-1}h)$$
$$+ f_1(\overline{m+1}h)], \qquad 1 \leqslant m \leqslant M-1 \tag{6.11}$$

Alternatively, if we use the truncated Taylor expansion

$$u_m^1 = u_m^0 + k \frac{\partial u_m^0}{\partial t} + \frac{1}{2} k^2 \frac{\partial^2 u_m^0}{\partial t^2}$$

and replace the first and second derivatives by their values from (6.2) and (6.1) then also we get (6.11).

The boundary conditions (6.3) become

$$u_0^n = g_1^n \text{ and } u_M^n = g_2^n, \; n = 1, 2, \ldots \tag{6.12}$$

Formula (6.7) may now be used to advance the computation for $n \geqslant 1$. Replacing u_m^n by $\xi^n e^{i\beta mh}$, we get

$$\xi + \frac{1}{\xi} = 2 - 4p^2 \sin^2\left(\frac{\beta h}{2}\right) \tag{6.13}$$

$$\xi^2 - 2A\xi + 1 = 0 \tag{6.14}$$

where $$A = 1 - 2p^2 \sin^2\left(\frac{\beta h}{2}\right)$$

The solution of (6.14) is given by

$$\xi_1 = A + \sqrt{(A^2-1)}, \; \xi_2 = A - \sqrt{(A^2-1)}$$

From (6.14), we find that $\xi_1 = 1/\xi_2$.

If $A > 1$, $|\xi_1| > 1$; if $A < -1$, $|\xi_2| > 1$; if $|A| \leqslant 1$, $|\xi_1| = |\xi_2| = 1$. Thus, for stability, $-1 \leqslant A \leqslant 1$ or $-1 \leqslant 1 - 2p^2 \sin^2(\beta h/2) \leqslant 1$. Hence, we get $p \leqslant 1$. This is the criterion for stability.

We now consider the interpretation of the stability requirements. The analytic solution $u(x, t)$ of the differential equation (6.1) subject to the initial conditions

$$u(x, 0) = f(x), \; \frac{\partial u(x, 0)}{\partial t} = g(x), \; -\infty < x < \infty, \tag{6.15}$$

can be obtained in the form

$$u(x, t) = \frac{1}{2} [f(x-t) + f(x+t)] + \frac{1}{2} \int_{x-t}^{x+t} g(\eta)\, d\eta \tag{6.16}$$

This is called the *d'Alembert solution.*

The lines $x-t$ = constant and $x+t$ = constant are the characteristics of the differential equation (6.1). The value of $u(x, t)$ at the nodal point (x_m, t_n) is obtained from (6.16) as

$$u(x_m, t_n) = \frac{1}{2}[f(x_m-t_n)+f(x_m+t_n)]+\frac{1}{2}\int_{x_m-t_n}^{x_m+t_n} g(\eta)\, d\eta \quad (6.17)$$

It follows from (6.17) that the value $u(x_m, t_n)$ is determined by prescribed values of $f(x)$ at the end points of the interval (x_m-t_n, x_m+t_n) and by prescribed values of $g(x)$ on that interval. The characteristics through the nodal point (x_m, t_n) are given by

$$\begin{aligned} x-t &= x_m-t_n \\ x+t &= x_m+t_n \end{aligned} \quad (6.18)$$

which meet the initial line $t = 0$ at the points $x = x_m-t_n$ and $x = x_m+t_n$. The interval (x_m-t_n, x_m+t_n) is finite and of length $2t_n$. The characteristics are sketched in Figure 6.1. The triange PAB is called the *region of determination* of the analytic solution at the nodal point (x_m, t_n). Thus, we find that the initial data outside the interval (x_m-t_n, x_m+t_n) does not influence the solution at the nodal point (x_m, t_n).

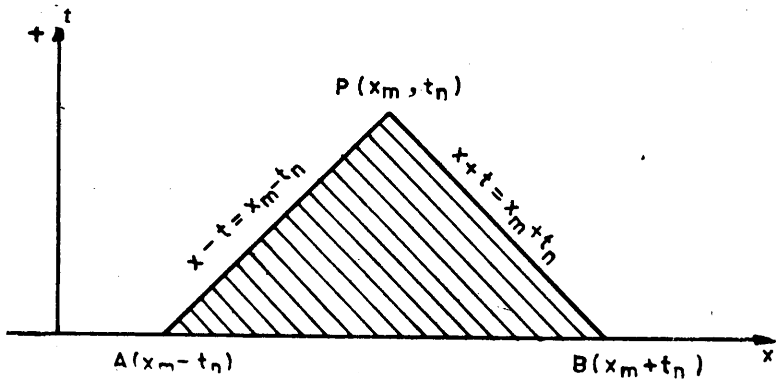

Fig. 6.1 Characteristics for the wave equation

We now examine the region of determination of the approximate solution u_m^n obtained with the help of the difference scheme (6.7). The lines through the nodal point (x_m, t_n) with slopes $\pm p$ play the role of the finite-difference characteristics. The slopes of these lines depend only on the choice of h and k. The finite-difference characteristics are given by

$$t-px = t_n-px_m \qquad t+px = t_n+px_m \quad (6.19)$$

which meet the initial line $t = 0$ at the points $x = x_m-t_n/p$ and $x=x_m+t_n/p$. Consequently, the approximate solution u_m^n depends upon

Thus, the difference scheme (6.21) will be stable if

(i) $\sigma \leqslant \frac{1}{4}$, $\tau \geqslant \frac{1}{4}$, $p > 0$,

(ii) $\sigma < \frac{1}{4}$, $\tau < \frac{1}{4}$, $0 < p^2 \leqslant \left(\frac{1-4\sigma}{1-4\tau}\right)$ (6.27)

When $\sigma = 0$, the above conditions reduce to the von Neumann conditions.

We now use (6.22) and (6.27) to determine accurate and stable methods. We find that the values $\tau = 1/4$, $\sigma \leqslant 1/4$ give unconditionally stable methods

$$\left[1+\left(\sigma-\frac{1}{4}p^2\right)\delta_x^2\right]\delta_t^2 u_m^n = p^2\delta_x^2 u_m^n \tag{6.28}$$

of order (k^2+h^2), and if $\sigma = 0$, then the *von Neumann* method is obtained. The values $\sigma = \tau = 1/12$ give high accuracy method

$$\left[1+\frac{1}{12}(1-p^2)\delta_x^2\right]\delta_t^2 u_m^n = p^2\delta_x^2 u_m^n \tag{6.29}$$

of $0(k^4+h^4)$ which is stable for $0 < p \leqslant 1$.

The solution of (6.1) with the help of the implicit schemes will require at each time step the solution of a tridiagonal system of linear algebraic equations. The values on the line $t = k$ can be obtained by using (6.11).

Example 6.1 Solve the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = \sin \pi x, \quad \frac{\partial u}{\partial t}(x, 0) = 0 \, . \qquad 0 \leqslant x \leqslant 1$$

$$u(0, t) = 0, \qquad u(1, t) = 0 \qquad t \geqslant 0$$

using the following methods

(a) $\delta_t^2 u_m^n = p^2\delta_x^2 u_m^n$

(b) $\left(1-\frac{p^2}{2}\delta_x^2\right)\delta_t^2 u_m^n = p^2\delta_x^2 u_m^n$

with $h=\frac{1}{3}$ and $p = \frac{1}{2}$.

The nodal points are

$$x_m = mh, \; t_n = nk, \quad 0 \leqslant m \leqslant 3, \; n = 0, 1, 2, \ldots$$

The difference method (a) for $p = \frac{1}{2}$ becomes

$$u_m^{n+1} = -u_m^{n-1} + \frac{1}{4}\left(u_{m-1}^n+6u_m^n+u_{m+1}^n\right)$$

The initial and boundary conditions become

$$u_m^0 = \sin \pi mh \text{ or } u_1^0 = \frac{\sqrt{3}}{2},\ u_2^0 = \frac{\sqrt{3}}{2}$$

$$\frac{\partial u_m^0}{\partial t} = 0 \text{ or } \frac{u_m^1 - u_m^{-1}}{2k} = 0,\ u_m^{-1} = u_m^1 \quad 1 \leqslant m \leqslant 2$$

$$u_0^n = 0,\ u_3^n = 0,\ n = 0, 1, 2, \cdots$$

We have

$$n = 0,\ u_m^1 = -u_m^{-1} + \frac{1}{4}\,(u_{m-1}^0 + 6u_m^0 + u_{m+1}^0) \quad 1 \leqslant m \leqslant 2$$

$$m = 1,\ u_1^1 = -u_1^{-1} + \frac{1}{4}\,(u_0^0 + 6u_1^0 + u_2^0)$$

or $$u_1^1 = \frac{1}{8}\,(u_0^0 + 6u_1^0 + u_2^0)$$
$$= 0.7578$$

$$m = 2,\ u_2^1 = -u_2^{-1} + \frac{1}{4}\,(u_1^0 + 6u_2^0 + u_3^0)$$

or $$u_2^1 = \frac{1}{8}\,(u_1^0 + 6u_2^0 + u_3^0)$$
$$= 0.7578$$

$$n = 1,\ u_m^2 = -u_m^0 + \frac{1}{4}\,(u_{m-1}^1 + 6u_m^1 + u_{m+1}^1) \quad 1 \leqslant m \leqslant 2$$

$$m = 1,\ u_1^2 = -u_1^0 + \frac{1}{4}\,(u_0^1 + 6u_1^1 + u_2^1)$$
$$= 0.4601$$

$$m = 2,\ u_2^2 = -u_2^0 + \frac{1}{4}\,(u_1^1 + 6u_2^1 + u_3^1)$$
$$= 0.4601$$

The method (b) for $p = \frac{1}{2}$ becomes

$$\left[1 - \frac{1}{8}\,\delta_x^2\right]\delta_t^2\, u_m^n = \frac{1}{4}\,\delta_x^2\, u_m^n$$

or $$-u_{m-1}^{n+1} + 10u_m^{n+1} - u_{m+1}^{n+1} = -10u_m^{n-1} + 16u_m^n + (u_{m-1}^{n-1} + u_{m+1}^{n-1})$$

This is a 2-step method and we require the solution values at $t = k$ to start the computation.

We have

$$n = 1,\ -u_{m-1}^2 + 10u_m^2 - u_{m+1}^2 = -10u_m^0 + 16u_m^1 + (u_{m-1}^0 + u_{m+1}^0) \quad 1 \leqslant m \leqslant 2$$

$$m = 1,\ -u_0^2 + 10u_1^2 - u_2^2 = -10u_1^0 + 16u_1^1 + (u_0^0 + u_2^0)$$

$$m = 2,\ -u_1^2 + 10u_2^2 - u_3^2 = -10u_2^0 + 16u_2^1 + (u_1^0 + u_3^0)$$

which may be expressed in the form

$$u_{l,m}^{n+1} = 2(1-2p^2)u_{l,m}^n+p^2[u_{l+1,m}^n+u_{l-1,m}^n+u_{l,m+1}^n+u_{l,m-1}^n]-u_{l,m}^{n-1} \quad (6.34)$$

It is easily verified that the truncation error of (6.34) is given by

$$T_{l,m}^n = \frac{1}{12}k^2h^2\left[(p^2-1)\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)+2p^2\frac{\partial^4 u}{\partial x^2\partial y^2}\right]+\ldots \quad (6.35)$$

which is of order (k^2+h^2).

To examine the stability of (6.34), we consider the solution of the form

$$u_{l,m}^n = \exp(i\mu nk)\exp(i\theta_1 lh)\exp(i\varphi_1 mh) \quad (6.36)$$

where θ_1 and φ_1 are arbitrary real numbers and μ may be a complex parameter. We put $\lambda = \mu k$, $\theta = \theta_1 h$, $\varphi = \varphi_1 h$.

Substituting (6.36) into (6.34) and simplifying the result we get

$$\sin^2\frac{\lambda}{2} = p^2\left(\sin^2\frac{\theta}{2}+\sin^2\frac{\varphi}{2}\right) \quad (6.37)$$

Since, $0 \leqslant \sin^2\varphi/2, \sin^2\theta/2 \leqslant 1$, the method (6.34) will be stable if, $0 < p \leqslant 1/\sqrt{2}$.

The second initial condition in (6.32) may be used to find the values on the plane $t = k$ in a manner as obtained in (6.11).

6.3.2 Implicit difference schemes

A simple difference equation of (6.31) based on Padé approximation to (6.33) can be written as

$$(1+\tau\delta_t^2)^{-1}\delta_t^2 u_{l,m}^n = p^2[(1+\sigma\delta_x^2)^{-1}\delta_x^2+(1+\sigma\delta_y^2)^{-1}\delta_y^2]\,u_{l,m}^n \quad (6.38)$$

The order accuracy of (6.38) is (k^2+h^2) and τ, σ are arbitrary parameters. Simplifying (6.38), we get

$$[(1+\sigma\delta_x^2)(1+\sigma\delta_y^2)-\tau p^2(\delta_x^2+\delta_y^2+2\sigma\delta_x^2\delta_y^2)]\delta_t^2 u_{l,m}^n$$
$$= p^2(\delta_x^2+\delta_y^2+2\sigma\delta_x^2\delta_y^2)u_{l,m}^n \quad (6.39)$$

The above scheme can also be written as

$$[1+(\sigma-\tau p^2)\,\delta_x^2][1+(\sigma-\tau p^2)\delta_y^2]\,\delta_t^2 u_{l,m}^n = p^2(\delta_x^2+\delta_y^2+2\sigma\delta_x^2\delta_y^2)\,u_{l,m}^n \quad (6.40)$$

The added term $\tau^2p^4\delta_x^2\delta_y^2\delta_t^2u_{l,m}^n$ is of higher order and does not alter the accuracy but enables a factorization of the operator on the left-hand side of (6.39). In order to determine the accuracy of the two parameter family of difference scheme (6.40), the Taylor expansion of the terms on both sides of (6.40) is carried out about the reference node (lh, mh, nk). If the resulting terms on the left-hand side are subtracted from those on the right hand side, the truncation error is found to be

$$T_{l,m}^n = h^2k^2\left[\left(\frac{1}{12}-\sigma\right)\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)_{l,m}^n+\left(\tau-\frac{1}{12}\right)p^2\left(\frac{\partial^4 u}{\partial t^4}\right)_{l,m}^n\right]+\ldots \quad (6.41)$$

where we have used the relations

$$\delta_t^2 u\,(lh, mh, nk) = p^2 C_0^{(2)} + \frac{1}{12} p^4 (C_0^{(4)} + 2C_4^{(0)}) + \frac{1}{360} p^6 (C_0^{(6)} + 3C_4^{(2)}) + \ldots$$

$$(\delta_x^2 + \delta_y^2) u\,(lh, mh, nk) = C_0^{(2)} + \frac{1}{12} C_0^{(4)} + \frac{1}{360} C_0^{(6)} + \ldots$$

$$\delta_x^2 \delta_y^2 u(lh, mh, nk) = C_4^{(0)} + \frac{1}{12} C_4^{(2)} + \ldots$$

$$(\delta_x^2 + \delta_y^2)\, \delta_t^2\, u\,(lh, mh, nk) = p^2 (C_0^{(4)} + 2C_4^{(0)}) + \frac{1}{12} p^2 \left((p^2+1)\, C_0^{(6)} + (1+3p^2) C_4^{(2)}\right) + \ldots$$

$$\delta_x^2 \delta_y^2 \delta_t^2\, u\,(lh, mh, nk) = p^2 C_4^{(2)} + \ldots \tag{6.42}$$

and

$$C_0^{(2)} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{l,m}^{n}$$

$$C_0^{(4)} = h^4 \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right)_{l,m}^{n}$$

$$C_0^{(6)} = h^6 \left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} \right)_{l,m}^{n}$$

$$C_4^{(0)} = h^4 \left(\frac{\partial^4}{\partial x^2\, \partial y^2} \right)_{l,m}^{n}$$

$$C_4^{(2)} = h^6 \left[\frac{\partial^4}{\partial x^2\, \partial y^2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right]_{l,m}^{n}$$

Thus all members of the family of difference scheme (6.40) for arbitrary τ and σ have an order of accuracy (k^2+h^2). The parameters τ and σ should be so chosen that difference scheme (6.40) is not only accurate but also stable. The method of von Neumann can be applied to examine the stability of (6.40). Substituting (6.36) into (6.40), we find that the difference scheme will be stable if

$$0 < p^2 \left[\frac{\sin^2 \frac{\theta}{2} + \sin^2 \frac{\varphi}{2} - 8\sigma \sin^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2}}{\left(1 - 4(\sigma - \tau p^2) \sin^2 \frac{\theta}{2}\right)\left(1 - 4(\sigma - \tau p^2) \sin^2 \frac{\varphi}{2}\right)} \right] \leqslant 1 \tag{6.43}$$

where $0 \leqslant \sin^2 \theta/2, \sin^2 \varphi/2 \leqslant 1$. This is satisfied if

$$0 < \frac{2p^2(1-4\sigma)}{[(1-4\sigma)+4\tau p^2)]^2} \leqslant 1$$

and

(i) $[1+(\sigma-\tau p^2)\,\delta_x^2]\,\overset{*}{u}{}_{l,m}^{n+1} = \dfrac{p^2}{\sigma-\tau p^2}\,[-1+((\sigma-\tau p^2)-2\sigma)\,\delta_y^2]u_{l,m}^n$

(ii) $[1+(\sigma-\tau p^2)\,\delta_y^2]\,(u_{l,m}^{n+1}-2u_{l,m}^n+u_{l,m}^{n-1})$

$$= \overset{*}{u}{}_{l,m}^{n+1}+\frac{p^2}{\sigma-\tau p^2}\,(1+2\sigma\delta_y^2)\,u_{l,m}^n \qquad (6.54)$$

In all the split formulas, the intermediate boundary conditions must be obtained explicitly from the second formula in order to maintain the accuracy of the difference schemes. For example, in (6.54) the intermediate boundary conditions are given by

$$\overset{*}{u}{}_{l,m}^{n+1} = (1+(\sigma-\tau p^2)\,\delta_y^2)\,(g_{l,m}^{n+1}-2g_{l,m}^n+g_{l,m}^{n-1})-\frac{p^2}{\sigma-\tau p^2}\,(1+2\sigma\delta_y^2)\,g_{l,m}^n \qquad (6.55)$$

where the appropriate values $g_{l,m}^n$ are obtained from (6.32).

Example 6.2 Solve the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

$$u(x, y, 0) = \sin \pi x \sin \pi y$$

$$\frac{\partial u(x, y, 0)}{\partial t} = 0 \qquad 0 \leqslant x, y \leqslant 1$$

$$u = 0$$

on the boundary of the unit square for $t > 0$, using the Lees split

$$\left(1-\frac{1}{4}p^2\delta_x^2\right)(\overset{*}{u}{}_{l,m}^{n+1}-2u_{l,m}^n+u_{l,m}^{n-1}) = p^2\,(\delta_x^2+\delta_y^2)\,u_{l,m}^n$$

$$\left(1-\frac{1}{4}p^2\delta_y^2\right)u_{l,m}^{n+1} = \overset{*}{u}{}_{l,m}^{n+1}-\frac{1}{4}p^2\delta_y^2\,(2u_{l,m}^n-u_{l,m}^{n-1})$$

with $h = \frac{1}{3}$ and $p = \frac{1}{2}$.

The nodal points are

$$x_l = lh,\ y_m = mh,\ t_n = nk, \qquad 0 \leqslant l, m \leqslant 3,\ n = 0, 1, 2, \ldots$$

The initial and boundary conditions become

$$u_{l,m}^0 = \sin \pi lh \sin \pi mh, \quad \frac{u_{l,m}^1-u_{l,m}^{-1}}{2k} = 0$$

$$u_{0,m}^n = u_{l,0}^n = 0 \qquad 0 \leqslant l, m \leqslant 3$$

The Lees method for $p = \frac{1}{2}$ may be written as

$$-\overset{*}{u}{}_{l-1,m}^{n+1}+18\overset{*}{u}{}_{l,m}^{n+1}-\overset{*}{u}{}_{l+1,m}^{n+1} = -16u_{l,m}^{n-1}+\delta_x^2\,u_{l,m}^{n-1}+32u_{l,m}^n-2\delta_x^2\,u_{l,m}^n$$
$$+4(\delta_x^2+\delta_y^2)u_{l,m}^n$$

$$-u_{l,m-1}^{n+1}+18u_{l,m}^{n+1}-u_{l,m+1}^{n+1} = 16\overset{*}{u}_{l,m}^{n+1}-2\delta_y^2 u_{l,m}^{n}+\delta_y^2 u_{l,m}^{n-1}$$

$$\overset{*}{u}_{0,m}^{n+1} = 0,\ \overset{*}{u}_{l,0}^{n+1} = 0 \qquad 1 \leqslant l, m \leqslant 2$$

The solution values at the level $t = k$ may be determined from the equation

$$u_{l,m}^1 = u_{l,m}^0+k\frac{\partial u_{l,m}^0}{\partial t}+\frac{1}{2}k^2\frac{\partial^2 u_{l,m}^0}{\partial t^2}+0(k^3)$$

or

$$u_{l,m}^1 = u_{l,m}^0+\frac{1}{2}p^2(\delta_x^2+\delta_y^2)\,u_{l,m}^0 \qquad 1 \leqslant l, m \leqslant 2$$

We have

$$u_{l,m}^0 = \sin \pi lh \sin \pi mh \qquad 1 \leqslant l, m \leqslant 2$$

$$u_{1,1}^0 = \frac{3}{4},\ u_{2,1}^0 = \frac{3}{4}$$

$$u_{2,1}^0 = \frac{3}{4},\ u_{2,2}^0 = \frac{3}{4}$$

$$u_{l,m}^1 = u_{l,m}^0+\frac{1}{8}\,(\delta_x^2+\delta_y^2)\,u_{l,m}^0 \qquad 1 \leqslant l, m \leqslant 2$$

$$u_{1,1}^1 = u_{1,1}^0+\frac{1}{8}\,(u_{0,1}^0+u_{2,1}^0+u_{1,2}^0+u_{1,0}^0-4u_{1,1}^0)$$

$$u_{1,1}^1 = \frac{9}{16}$$

$$u_{2,1}^1 = \frac{9}{16},\ u_{1,2}^1 = \frac{9}{16},\ u_{2,2}^1 = \frac{9}{16}$$

$$n = 1, -\overset{*}{u}_{l-1,m}^2+18\overset{*}{u}_{l,m}^2-\overset{*}{u}_{l+1,m}^2$$

$$= -16u_{l,m}^0+\delta_x^2 u_{l,m}^0+32u_{l,m}^1-2\delta_x^2 u_{l,m}^1+4(\delta_x^2+\delta_y^2)u_{l,m}^1$$

$$l = 1, m = 1, -\overset{*}{u}_{0,1}^2+18\overset{*}{u}_{1,1}^2-\overset{*}{u}_{2,1}^2$$

$$= -16u_{1,1}^0+\delta_x^2 u_{1,1}^0+32u_{1,1}^1-2\delta_x^2 u_{1,1}^1+4(\delta_x^2+\delta_y^2)\,u_{1,1}^1$$

$$18\overset{*}{u}_{1,1}^2-\overset{*}{u}_{2,1}^2 = \frac{15}{8}$$

$$l = 2, m = 1, -\overset{*}{u}_{1,1}^2+18\overset{*}{u}_{2,1}^2-\overset{*}{u}_{3,1}^2$$

$$= -16u_{2,1}^0+\delta_x^2 u_{2,1}^0+32u_{2,1}^1-2\delta_x^2 u_{2,1}^1+4(\delta_x^2+\delta_y^2)u_{2,1}^1$$

$$-\overset{*}{u}_{1,1}^2+18\overset{*}{u}_{2,1}^2 = \frac{15}{8}$$

$$\overset{*}{u}_{1,1}^2 = \frac{15}{136},\ \overset{*}{u}_{2,1}^2 = \frac{15}{136}$$

$$\overset{*}{u}_{1,2}^2 = \frac{15}{136},\ \overset{*}{u}_{2,2}^2 = \frac{15}{136}$$

$$-u^2_{l,m-1}+18u^2_{l,m}-u^2_{l,m+1}$$
$$= 16u^{*2}_{l,m}-2\delta_y^2 u^1_{l,m}+\delta_y^2 u^0_{l,m} \qquad 1 \leqslant l, m \leqslant 2$$
$$l = 1, m = 1, -u^2_{1,0}+18u^2_{1,1}-u^2_{1,2} = 16u^{*2}_{1,1}-2\delta_y^2 u^1_{1,1}+\delta_y^2 u^0_{1,1}$$
$$18u^2_{1,1}-u^2_{1,2} = \frac{291}{136}$$
$$l = 1, m = 2, -u^2_{1,1}+18u^2_{1,2}-u^2_{1,3} = 16u^{*2}_{1,2}-2\delta_y^2 u^1_{1,2}+\delta_y^2 u^0_{1,2}$$
$$-u^2_{1,1}+18u^2_{1,2} = \frac{291}{136}$$
$$u^2_{1,1} = \frac{291}{2312} = 0.1259$$
$$u^2_{1,2} = \frac{291}{2312} = 0.1259$$
$$u^2_{2,1} = \frac{291}{2312} = 0.1259$$
$$u^2_{2,2} = \frac{291}{2312} = 0.1259$$

6.3.4 Results from computation

We consider Equation (6.31) together with the initial conditions

$$u(x, y, 0) = \sin \pi x \sin \pi y$$
$$\frac{\partial u}{\partial t} = 0 \quad \text{for } 0 \leqslant x, y \leqslant 1, t = 0 \tag{6.56}$$

and the boundary condition

$$u = 0$$

on the boundary of the unit square for $t > 0$. The theoretical solution of this problem is

$$u(x, y, t) = \sin \pi x \sin \pi y \cos \sqrt{2}\pi t$$

We take $h = 1/11$ and $k = p/11$, where p satisfies the stability condition. Using the Lees split (6.51) and choosing the values of σ and τ from Figure 6.3, we compute the numerical solution. The error values (the difference between the numerical solution and the theoretical solution) at one of the four grid points nearest to the centre of the unit square are set out in Table 6.2. We find that the difference scheme corresponding to $\sigma = \tau = 1/8$ gives less error in comparison to other difference schemes of accuracy of $0(k^2+h^2)$. The Fairweather-Mitchell formula which is of accuracy (h^4+k^4) gives smaller error for all values of h and k satisfying the stability condition $0 < p < \sqrt{3}-1$. For the values of h and k which violate the stability condition, the Fairweather-Mitchell formula gives poor results after certain value of t whereas the scheme corresponding to $\sigma = \tau = 1/8$ gives stable and accurate results.

TABLE 6.2 The Error Values in the Solution of Problem (6.31) Subject to (6.56)
(all digits are to be Multiplied by 10^{-2})

	$O(h^2+k^2)$			$O(h^4+k^2)$			$O(h^4+k^4)$
σ	1/8	0	0	1/12	1/12	1/12	1/12
τ	1/8	1/4	1/2	1/8	1/4	1/2	1/12
$t \backslash k$	$k = 0.06,\quad p = 0.66$						
0.6	+ .0221	− 1.0148	− 2.0010	− .1610	−0.6450	− 1.6276	−.0008
1.2	− .0831	+ 3.7914	+ 7.4418	+ .6039	+2.4139	+ 6.0638	+.0031
1.8	+ .1542	− 6.9506	+13.4848	−1.1181	+4.4444	−11.0367	−.0058
2.4	− .1981	+ 8.7278	+16.5541	+1.4309	+5.6272	+13.6664	+.0075
3.0	+ .1820	− 7.6253	−13.7287	−1.3036	−5.0081	−11.5656	−.0068
$t \backslash k$	$k = 0.075,\quad p = 0.825$						
0.6	−.0656	− 1.3432	− 2.8557	− .2438	− .9830	− 2.4918	+.0001
1.2	+ .2498	+ 5.0894	+10.7493	+0.9279	+3.7302	+ 9.3945	−.0005
1.8	− .4651	− 9.3354	−19.3590	−1.7243	−6.8717	−16.9949	−.0004
2.4	+ .5977	+11.6335	+23.3051	+2.2073	+8.6499	+20.6421	−.2775
3.0	− .5477	−10.0070	−18.3487	−2.0051	−7.5703	−16.6136	−60.9821
$t \backslash k$	$k = 0.1,\quad p = 1.1$						
0.6	− .2417	− 2.0182	− 4.6134	− .4117	− 1.6751	− 4.2689	+.0034
1.2	+ .9448	+ 7.8420	+17.7486	+1.6081	+ 6.5165	+16.4475	+.0138
1.8	−1.7694	−17.4374	−31.4663	−3.0058	−11.9953	−29.2903	−.5955
2.4	+2.2745	+17.6927	+36.1684	+3.8465	+14.8910	+33.9894	−715.3420
3.0	−2.0470	−14.5952	−24.9436	−3.4701	−12.5355	−24.0789	−8493.75

6.4 DIFFERENCE SCHEMES FOR EQUATIONS IN THREE SPACE VARIABLES WITH CONSTANT COEFFICIENTS

We derive difference schemes for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{6.57}$$

subject to the initial and boundary conditions

$$u(x, y, z, 0) = f_1(x, y, z)$$

$$\frac{\partial u(x, y, z, 0)}{\partial t} = f_2(x, y, z),\ (x, y, z) \in \mathcal{R}$$

$$u(x, y, z, t) = g(x, y, z, t),$$

$$(x, y, z, t) \in \partial\mathcal{R} \times [0 \leqslant t \leqslant T] \tag{6.58}$$

where $\partial\mathcal{R}$ is the boundary of

$$\mathcal{R} = [0 \leqslant x, y, z \leqslant 1]$$

We assume that the nodal points in $\mathcal{R}$ are given by (l_1h, l_2h, l_3h, nk), $l_1, l_2, l_3 = 0, 1, 2, \ldots, M, n = 0, 1, 2, \ldots, N$.

The difference equation for (6.57) based on Padé approximation can be written as

$$(1+\tau\delta_t^2)^{-1}\delta_t^2 u^n = p^2[(1+\sigma\delta_x^2)^{-1}\delta_x^2 + (1+\sigma\delta_y^2)^{-1}\delta_y^2 + (1+\sigma\delta_z^2)^{-1}\delta_z^2]u^n \tag{6.59}$$

where u^n stands for $u^n_{l_1,l_2,l_3}$ which is the approximate value of u at $(l_1h,\ l_2h,\ l_3h,\ nk)$.

The order of accuracy of (6.59) is of $0(k^2+h^2)$ for arbitrary σ and τ and it increases to $0(k^4+h^4)$ for $\sigma = \tau = 1/12$. Simplifying (6.59), we obtain

$$[1+(\sigma-\tau p^2)\textstyle\sum\delta_x^2+(\sigma^2-2\sigma\tau p^2)\sum\delta_x^2\delta_y^2+(\sigma^3-3\sigma^2\tau p^2)\Pi\delta_x^2]\delta_t^2 u^n$$

$$= p^2[\textstyle\sum\delta_x^2+2\sigma\sum\delta_x^2\delta_y^2+3\sigma^2\Pi\delta_x^2]u^n \tag{6.60}$$

where

$$\textstyle\sum\delta_x^2 = \delta_x^2+\delta_y^2+\delta_z^2,$$

$$\textstyle\sum\delta_x^2\delta_y^2 = \delta_x^2\delta_y^2+\delta_y^2\delta_z^2+\delta_z^2\delta_x^2,$$

and

$$\Pi\delta_x^2 = \delta_x^2\delta_y^2\delta_z^2$$

The difference scheme (6.60) to the order of accuracy of (k^2+h^2) for arbitrary σ and τ with the factorized operators on the left side can be written in the form

$$(\Pi[1+(\sigma-\tau p^2)\delta_x^2])\delta_t^2 u^n = p^2[\textstyle\sum\delta_x^2+2\sigma\sum\delta_x^2\delta_y^2+3\sigma^2\Pi\delta_x^2]u^n \tag{6.61}$$

We may use the von Neumann method to show that the difference scheme (6.61) is unconditionally stable for $\sigma < 1/4$, $\tau > 1/8$. For $\tau > 1/8$, $\sigma = 0$ and $\tau = 1/4$, σ arbitrary, we may write (6.61) as

$$(\Pi[1-\tau p^2\delta_x^2])\delta_t^2 u^n = p^2(\sum \delta_x^2)u^n \tag{6.62}$$

and

$$\Pi\left[1+\left[\sigma-\frac{1}{4}p^2\right]\delta_x^2\right](u^{n+1}+u^{n-1}) = 2\Pi\left\{1+\left(\sigma+\frac{1}{4}p^2\right)\delta_x^2\right\}u^n \tag{6.63}$$

respectively. From application view point, it is computationally economical to write (6.61) as an ADI scheme. The *D' Yakonov* splitting is given by

(i) $[1+(\sigma-\tau p^2)\delta_x^2]u^{*n+1} = p^2[\sum\delta_x^2+2\sigma\sum\delta_x^2\delta_y^2+3\sigma^2\Pi\delta_x^2]u^n$
$\qquad +\Pi[1+(\sigma-\tau p^2)\delta_x^2](2u^n-u^{n-1})$

(ii) $[1+(\sigma-\tau p^2)\delta_y^2]u^{**n+1} = u^{*n+1}$

(iii) $[1+(\sigma-\tau p^2)\delta_z^2]u^{n+1} = u^{**n+1}$ (6.64)

where u^{*n+1} and u^{**n+1} are intermediate values. The *Lees* split for (6.62) can be written as

(i) $[1-\tau p^2\delta_x^2]u^{*n+1} = p^2\sum\delta_x^2 u^n$

(ii) $[1-\tau p^2\delta_y^2]u^{**n+1} = u^{*n+1}$

(iii) $[1-\tau p^2\delta_z^2](u^{n+1}-2u^n+u^{n-1}) = u^{**n+1}$ (6.65)

Another possible split of (6.62) is of the form

(i) $[1-\tau p^2\delta_x^2]\, u^{*n+1} = \left[\frac{1}{\tau}+p^2(\delta_y^2+\delta_z^2)\right]u^n$

(ii) $[1-\tau p^2\delta_y^2]u^{**n+1} = u^{*n+1}-p^2\delta_y^2\, u^n$

(iii) $[1-\tau p^2\delta_z^2](u^{n+1}-2u^n+u^{n-1}) = u^{**n+1}-\frac{1}{\tau}\, u^n$ (6.66)

The difference scheme (6.63) can also be written as

(i) $\left[1+\left(\sigma-\frac{1}{4}p^2\right)\delta_x^2\right]u^{*n+1}$
$\qquad = 2\left[1+\left(\sigma+\frac{1}{4}p^2\right)\delta_y^2\right]\left[1+\left(\sigma+\frac{1}{4}p^2\right)\delta_z^2\right]u^n$

(ii) $\left[1+\left(\sigma-\frac{1}{4}p^2\right)\delta_y^2\right]u^{**n+1}$
$$= u^{*n+1}-\frac{\frac{1}{4}p^2+\sigma}{\frac{1}{4}p^2-\sigma}\left[1+\left(\sigma+\frac{1}{4}p^2\right)\delta_z^2\right]u^n$$

(iii) $\left[1+\left(\sigma-\frac{1}{4}p^2\right)\delta_z^2\right](u^{n+1}+u^{n-1})$

$$=\left[1+\frac{\frac{1}{4}p^2+\sigma}{\frac{1}{4}p^2-\sigma}\right]u^{**n+1}+\left[\frac{\frac{1}{4}p^2+\sigma}{\frac{1}{4}p^2-\sigma}\right]^2\left[1+\left(\frac{1}{4}p^2+\sigma\right)\delta_z^2\right]u^n \tag{6.67}$$

The intermediate boundary conditions can be determined again easily. For example, in (6.66) they are given by

(i) $u^{*n+1} = [1-\tau p^2\delta_y^2][1-\tau p^2\delta_z^2](g^{n+1}-2g^n+g^{n-1}) + \frac{1}{\tau}g^n$

(ii) $u^{**n+1} = [1-\tau p^2\delta_z^2](g^{n+1}-2g^n+g^{n-1})+ \frac{1}{\tau}g^n$ (6.68)

6.5 DIFFERENCE SCHEMES FOR EQUATIONS WITH VARIABLE COEFFICIENTS

6.5.1 One space dimension

Consider the linear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = a(x, t)\frac{\partial^2 u}{\partial x^2} \tag{6.69}$$

In the region $\mathcal{R} = [0 \leqslant x \leqslant 1]\times[0, T]$, subject to the initial and boundary conditions (6.2) and (6.3). We replace (6.69) at the nodal point (mh, nk) by the difference scheme

$$(1+\tau\delta_t^2)^{-1}\delta_t^2 u_m^n = p^2a_m^n(1+\sigma\delta_x^2)^{-1}\delta_x^2 u_m^n \tag{6.70}$$

where

$$a_m^n = a(x_m, t_n)$$

The approximation represented by (6.70) has truncation error of order (k^2+h^2) for arbitrary σ and τ and this order increases to (k^4+h^4) for $\sigma = \tau = 1/12$.

Multiplying (6.70) with $(1+\tau\delta_t^2)$ and simplifying, we obtain

$$\delta_t^2[1-\tau p^2a_m^n Q_x^{-1}\delta_x^2]u_m^n = p^2a_m^n Q_x^{-1}\delta_x^2 u_m^n \tag{6.71}$$

where

$$(1+\sigma\delta_x^2) = Q_x$$

which may also be written as

$$[1-\tau p^2a_m^{n+1} Q_x^{-1}\delta_x^2]u_m^{n+1}$$
$$= 2[1-\tau p^2a_m^n Q_x^{-1}\delta_x^2]u_m^n-[1-\tau p^2a_m^{n-1} Q_x^{-1}\delta_x^2]u_m^{n-1}+p^2a_m^n Q_x^{-1}\delta_x^2 u_m^n \tag{6.72}$$